OPTIMIZATION OF ONE-DIMENSIONAL FLOWS, WHEN ONE OF THE CHARACTERISTIC VELOCITIES PASSES CONTINUOUSLY THROUGH ZERO

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The present paper is concerned with the problems of optimal control of one-dimensional fluid flows, when one of the characteristic velocities (velocity of propagation of weak discontinuities passes through zero. We know [1] that the point at which the characteristic velocity changes its sign, is the singular point of the system of differential equations describing the flow. We find that the system of equations for Lagrange multipliers which is obtained in the course of optimizing such flows, also has a singularity at the point at which the characteristic velocity becomes zero. Consequently, a problem arises of obtaining an unambiguous choice of the Lagrange multipliers at the singularity.

Below we show that the Lagrange multipliers in the optimal solution should be continuous and bounded, when passing through the singularity. The results obtained are illustrated by an example dealing with optimization of a magnetogasdynamic electric power generator, where the flow takes place continuously under the velocity of sound conditions.

1. Let us consider an arbitrary, one-dimensional steady flow of a continuous medium. We shall assume that, within the parametric space under consideration, one of the characteristic velocities obtained from the corresponding equations for a unsteady flow, becomes zero at some point. In this case we may write the equations for the steady flow of a continuous medium in their normal form, as

$$y_1' = y_1^{-1} f(x, y_k, v_i), \qquad y_j' = g_j(x, y_k, v_i) \tag{1.1}$$
$$(i = 1, \ldots, m; \quad k = 1, \ldots, n; \quad j = 2, \ldots, n)$$

Here x is the coordinate coinciding with the direction of the velocity of flow, y_j denotes the flow parameters such as e.g. velocity, pressure, density of the fluid, induced magnetic field strength, concentration of the components of the medium, etc; y_1 is the parameter proportional to the characteristic velocity which, by definition, changes its sign (for the gasdynamic problems, it is convenient to choose $y_1 = M - 1$ where M is the Mach number) and v_i denote various controls such as, form of the channel, intensity of applied magnetic field, electric potential difference etc.

We assume that v_i are functions of x and a prime denotes derivatives with respect to x.

We have shown in [1] that the above form of the steady state solution is feasible. Suitable choice of the independent variables may lead to the form, where only one of the equations will contain a singularity in its right hand side, while the remaining equations will be finite at the singular point.

The singular point of (1.1) can be determined by the following conditions

$$f(x, y_k, v_i) = 0, \qquad y_1 = 0 \qquad (k = 1, \ldots, n; \quad i = 1, \ldots, m) \tag{1.2}$$

and it will lie on an $(n-1)$-dimensional surface in the $(n+1)$-dimensional variable space, where the variables are $x, y_1, \ldots, y_n$. In the neighborhood of this point the integral curves lie on a two-dimensional plane [1].

The character of the singularity can be found, in the usual manner, from the coefficients of the linear expansion of the right-hand sides of (1.1) near the singular point.

Secular equation defining the eigenvalues λ, has the form

$$\begin{vmatrix} f_{y_1} - \lambda & f_x + f_{y_j} g_j + f_{r_i} v_i' \\ 1 & -\lambda \end{vmatrix} = 0 \qquad \begin{pmatrix} i = 1, \ldots, m \\ j = 2, \ldots, n \end{pmatrix} \tag{1.3}$$

where f_x, f_{y_1}, f_{yj} and f_{vi} are the corresponding partial derivatives and repeated indices (i, j) denote summation. Relation (1.3) yields

$$\lambda_{1,2} = {}^1/_2 \left[f_{y_1} \pm \sqrt{f^2_{y_1} + 4 (f_x + f_{y_j} g_j + f_{r_i} v_i')} \right] \tag{1.4}$$

Characteristic direction cosines at the singularity are of the same magnitude as the eigenvalues

$$y_1' = \lambda_{1,\,2} \tag{1.5}$$

In the earlier paper [1] we discuss in detail, how the sign of λ_1 and λ_2 influences the character of the singularities. Here, we shall only concern ourselves with two types of singularities, near which the flow is stable [1]: a node with negative eigenvalues $(\lambda_1 < 0,\ \lambda_2 < 0)$ and a saddle point $(\lambda_1 > 0,\ \lambda_2 < 0)$.

Before integrating the system (1.1) over a finite segment $[a, b]$ of the x-axis, we must specify the boundary conditions. It follows from [2], that at $x = a$ the number of these conditions is equal to the number of the positive characteristic velocities, while at $a = b$, it equals to the number of the negative characteristic velocities at that point.

When the characteristic velocities maintain their sign over the considered interval of integration, then the total number of the left and right boundary conditions is equal to n.

If, on the other hand, the characteristic velocity (in this case y_1) changes its sign on passing through a node at some interior point $x^* \in [a, b]$, then we have an additional boundary condition at $x = b$, since $y_1 > 0$ when $x = a$ and $y_1 < 0$ when $x = b$.

Thus, when a singularity of the node type exists, Eqs. (1.1) have $(n+1)$ boundary conditions which can be written as

$$\varphi_i (a, y_{ka}) = 0 \quad (i = 1, \ldots, r), \qquad \psi_j (b, y_{kb}) = 0 \quad (j = r+1, \ldots, n+1)$$
$$(k = 1, \ldots, n) \tag{1.6}$$

Here r denotes the number of the positive characteristic velocities at $x = a$, $(n + 1 - r)$ is the number of the negative characteristic velocities at $x = b$, φ_i and ψ_j are some known functions, while y_{ka} and y_{kb} denote the values at the points a and b respectively.

We assume that the boundary conditions (1.6) are such, that the integral curve with negative y_1' reaches the node, otherwise the flow corresponding to the conditions (1.6) exhibits a shock wave.

When a solution passes through a saddle point with positive y_1', then for $x = a$ and for $x = b$, we have $(n-1)$ boundary for conditions, since $y_1 < 0$ when $x = \alpha$ and $y_1 > 0$ when $x = b$. These conditions can also be written in the form of (1.6), by putting $j = r + 1, \ldots, n - 1$. In this case (1.2) should be used as an additional relation, in order to obtain the solution.

When a solution passes through a saddle point with negative y_1', the number of the boundary conditions is $(n + 1)$ just as in the case of the node, and we have a corresponding set of solutions with discontinuities ([3] gives a good illustration of the above argument). To obtain a continuous solution passing through a saddle point with $y_1' < 0$ we must reject one condition on each side and demand, that (1.2) holds. Thus, when a solution passes through a saddle point, we have $(n-1)$ conditions at $x = a$ and $x = b$, and the condition (1.2).

2. Let us now find the optimal controls $v_i(x)$, i.e. such controls, that the functional

$$J = \int_a^b \Phi(x, y_k, v_i)\,dx \quad (k = 1, \ldots, n; \quad i = 1, \ldots, m) \tag{2.1}$$

assumes its maximum value and the quantities x, y_k and v_i are connected by the differential Eqs. (1.1).

Function $\Phi(x, y_k, v_i)$ is assumed known and continuous, together with its first order partial derivatives. Arbitrary, piecewise continuous functions with first order discontinuities at a finite number of points on the interval $[a, b]$ will be considered as admissible controls, and we shall also assume that the controls v_i will have constraints such as e.g.

$$|v_i| \leqslant V_i \qquad (i = 1, \ldots, m) \tag{2.2}$$

imposed on them. These constraints will either be connected with the boundaries of applicability of (1.1), or will relate to the technical structure of the control itself. The quantities V_i in (2.2) are known constants. Let us construct an auxilliary functional

$$I = \int_a^b \Big\{\Phi(x, y_k, v_i) + \mu_1\Big[y_1' - \frac{f(x, y_k, v_i)}{y_1}\Big] + \mu_j[y_j' - g_j(x, y_k, v_i)]\Big\}\,dx$$
$$(k = 1, \ldots, n;\ j = 2, \ldots, n;\ i = 1, \ldots, m) \tag{2.3}$$

Here μ_1 and μ_j are the variable Lagrange multipliers. Admissible variations of the functionals J and I coincide, since Eqs. (1.1) are satisfied.

We shall only consider two possible types of solutions of (1.1) in which the characteristic velocity y_1 changes its sign when the corresponding integral curve passes through the singularity of the (1) node type and (2) saddle point type. We assume that the character of the singular point does not change under the admissible variation.

Let us subdivide the interval of integration $[a, b]$ into segments over which the controls $v_i(x)$ are continuous, and into segments, over which $y_1(x)$ has a constant sign.

To obtain all the necessary conditions, we need only consider a single point of discontinuity of the controls v_i and a single point at which y_1 changes its sign. We shall assume that the ends of the interval $[a, b]$ are not fixed. Let us find the first variation of I, taking into account the fact that μ_1 and μ_j are continuous at the points $x = d$ where the controls become discontinuous, while $y_k'(x^*)$ $(k = 1, \ldots, n)$ are continuous at the singular point $x = x^*$

$$\delta I = \int_a^b \Big[\Big(\Phi_{v_i} - \frac{\mu_1 f_{v_i}}{y_1} - \mu_j g_{jv_i}\Big)\delta v_i + \Big(\Phi_{y_1} - \mu_1\frac{(y_1 f_{y_1} - f)}{y_1^2} - \mu_j g_{jy_1} - \mu_1'\Big)\delta y_1 +$$
$$+ (\Phi_{y_k} - \mu_1 f_{y_k}/y_1 - \mu_j g_{jy_k} - \mu_k')\,\delta y_k\Big]dx + \{\Phi(x_-^*) - \Phi(x_+^*) + \tag{2.4}$$
$$+ y_s'(x^*)[\mu_s(x_+^*) - \mu_s(x_-^*)]\}\,\delta x^* + \mu_1(x_-^*)\,\delta y_{1-}^* - \mu_1(x_+^*)\delta y_{1+}^* + [\mu_k(x_-^*) -$$
$$- \mu_k(x_+^*)]\,\delta y_k^* + \{\Phi(x_{d-}) - \Phi(x_{d+}) - \mu_s(x_d)[y_s'(x_{d-}) - y_s'(x_{d+})]\}\,\delta x_d +$$
$$+ \mu_s(b)\,\delta y_{sb} - \mu_s(a)\,\delta y_{sa} + [\Phi(b) - y_s'(b)\mu_s(b)]\,\delta x_b - [\Phi(a) - y_s'(a)\mu_s(a)]\,\delta x_a$$
$$(i = 1, \ldots, m;\ j,\ k = 2, \ldots, n;\ s = 1, \ldots, n)$$

Here $\Phi_{v_i}, \Phi_{y_1}, \Phi_{y_k}, f_{v_i} f_{y_1}, f_{y_k}, g_{jv_i}, g_{jy_1}$, and g_{jy_k} are the corresponding partial derivatives of Φ, f and g_j, the subscripts a, b and d refer to variations at $x = a$, $x = b$ and $x = d$, and the asterisk denotes the quantities at the singular point. Symbols $\Phi(x_-)$, $\mu_s(x_-)$, $y_s'(x_-)$ $[\Phi(x_+), \mu_s(x_+), y_s'(x_+)]$ indicate that the relevant quantities are computed to the left [right] of the corresponding x.

We shall choose the Lagrange multipliers in such a manner, that only the variations of controls remain in (2.4), and we shall show that it can be done for any solution of (1.1) considered here, by choosing μ_1 and μ_k such, that

$$\mu_1' = \Phi_{y_1} - \mu_1 \frac{y_1 f_{y_1} - f}{y_1^2} - \mu_j g_{j y_1}, \quad \mu_k' = \Phi_{y_k} - \mu_1 \frac{f_{y_k}}{y_1} - \mu_j g_{j y_k} \quad (j, k = 2, \ldots, n) \tag{2.5}$$

holds on the intervals over which the quantity y_1 does not change its sign.

To integrate (2.5), we must select the boundary conditions whose form will depend on the character of the solution of (1.1).

3. It follows from (2.5), that the equations for Lagrange multipliers have a singularity at the point where y_1 vanishes. This raises the problem of unambiguous selection of μ_1 and μ_j at the singularity. The feasibility of constructing a unique solution for the Lagrange multipliers can be inferred from the boundary conditions (1.6) for the system (1.1), from the terms outside the integral sign at the points a, b and x^* in the expression (2.4) and from the character of the singularity of the system (1.1) and (2.5).

Let us consider a solution of (1.1) passing through the node, when conditions (1.6) hold at the ends a and b of the segment. Varying the right hand sides of (1.6), we obtain a system of $(n + 1)$ linear algebraic equations in δy_{ka}, δx_a, δy_{kb} and $\delta x_b (k = 1, \ldots, n)$

$$\varphi_{ix}\delta x_a + \varphi_{iy_k}\delta y_{ka} = 0 \qquad (x = a, \ i = 1, \ldots, r)$$

$$\psi_{jx}\delta x_b + \psi_{jy_k}\delta y_{kb} = 0 \qquad (x = b, \ j = r + 1, \ldots, n + 1)$$

Solving this system for r values for δy_{ka} and $(n - r + 1)$ values of δy_{kb} and inserting the results into (2.4) we obtain, after reducing the like terms, $(n - r)$ arbitrary variations δy_{ka} for $x = a$ and $(r - 1)$ arbitrary variations δy_{kb} for $x = b$. Equating the coefficients preceding these variations to zero, we obtain $(n - 1)$ boundary conditions necessary for the integration of (2.5), $(n - r)$ of these conditions referring to $x = a$, and the remaining $(r - 1)$ conditions referring to $x = b$. Equating to zero the coefficients preceding δx_a and δx_b, we obtain the conditions defining the positions of the ends of the interval of integration. We can easily see, that the number of conditions is insufficient to integrate the system (2.5).

Let us now consider the solution of (1.1) passing through a saddle point. In this case the conditions (1.6) hold at a and b, but $j = r + 1, \ldots, n - 1$. Repeating the argument given above we obtain, for (2.5), $(n - r)$ conditions for $x = a$ and $(r + 1)$ conditions for $x = b$, i.e. together $(n + 1)$ conditions.

Let us now introduce another variable $\mu = \mu_1 / y_1$. Then (2.5) can be written as

$$\mu' = y_1^{-1} (\Phi_{y_1} - \mu f_{y_1} - \mu_j g_{j y_1}), \quad \mu_k' = \Phi_{y_k} - \mu f_{y_k} - \mu_j g_{j y_k} \quad (j, k = 2, \ldots, n) \tag{3.1}$$

We can easily see that the singularity of the system (1.1) and (3.1) is defined by

$$f(x, y_k, v_i) = 0, \qquad y_1 = 0, \qquad \Phi_{y_1} - \mu f_{y_1} - \mu_j g_{j y_1} = 0 \tag{3.2}$$

Using the old variables we would find, that the last equation in (3.2) is $\mu_1 = 0$.

Since the right-hand sides of equations for the derivatives y_j' and μ_j' $(j = 2, \ldots, n)$ are finite near the singularity we find, that only first Eqs. of (1.1) and (3.1) are essential for investigation of the character of the singularity. Introducing differentiation with respect to t, we can write these equations in the form of an independent system, and linearizing their right-hand sides near the singularity, we obtain

$$y_{1t} = f_{y_1}\Delta y_1 + (f_x + f_{v_i}v_i' + f_{y_j}g_j)\,\Delta x$$

$$x_t = \Delta y_1, \qquad \mu_t = \alpha\Delta y_1 + \beta\Delta x - f_{y_1}\Delta\mu$$

Where Δy_1, Δx and $\Delta\mu$ are the increments of the corresponding quantities near the singularity, while α and β are the coefficients of expansion of the numerator of the right-hand side of the equation for μ'. Numerically, α and β are not essential.

Eigenvalues of the singularity are, in this case, given by

$$[-\lambda\,(f_{y_1} - \lambda) - (f_x + f_{v_i}v_i' + f_{y_j}g_j)]\,(-f_{y_1} - \lambda) = 0 \tag{3.3}$$

from which we find that λ_1 and λ_2 are determined by (1.4), while

$$\lambda_3 = -f_{y_1} \tag{3.4}$$

Solution for Δy_1, Δx and $\Delta\mu$ has the following form near the singularity:

$$\Delta y_1 = +\,c_1\lambda_1 \exp\lambda_1 t + c_2\lambda_2 \exp\lambda_2 t, \qquad \Delta x = c_1 \exp\lambda_1 t + c_2 \exp\,\lambda_2 t$$

$$\Delta\mu = c_1\,(\beta - \lambda_1\alpha)\,(\lambda_1 + f_{y_1})^{-1} \exp\lambda_1 t + c_2\,(\beta - \lambda_2\alpha)(\lambda_2 + f_{y_1})^{-1} \exp\lambda_2 t + c_3 \exp\lambda_3 t$$

where c_1, c_2 and c_3 are arbitrary constants.

If the singularity of (1.1) is a node, then $\lambda_1 < 0$, $\lambda_2 < 0$ and $\lambda_1 > \lambda_2$. Then (3.4) yields $\lambda_3 > 0$.

On approaching the singularity ($t \to \infty$), we have

$$\Delta y_1 \sim \exp\lambda_1 t, \quad \Delta x \sim \exp\lambda_1 t, \quad \Delta\mu \sim \exp\lambda_3 t$$

Behavior of $\mu_1 = \mu y_1$ is governed by the term $\exp\,(\lambda_1 + \lambda_3)t$. Since $\lambda_1 + \lambda_3 > 0$, we have $\mu_1 \to \infty$, when $t \to \infty$.

This means that the singularity given by (3.2) is a generalized saddle point and $\mu_1 = 0$ corresponds to continuous solutions passing through it.

If the system (1.1) has a saddle point as a singularity, then $\lambda_1 > 0$ and $\lambda_2 < 0$. Suppose that at the same time $f_{y_1} < 0$. Then (3.4) yields $\lambda_3 > 0$. In this case we have, on approaching the singularity as $t \to \infty$,

$$\Delta y_1 \sim \exp\lambda_2 t, \qquad \Delta x \sim \exp\lambda_2 t, \qquad \Delta\mu \sim \exp\lambda_3 t \qquad (c_1 = 0)$$

Since the behavior of μ_1 is governed by the term $\exp\,(\lambda_2 + \lambda_3)t$, $\lambda_2 + \lambda_3 < 0$ implies that $\mu_1 \to 0$ as $t \to \infty$. On approaching the singularity as $t \to -\infty$, we have

$$\Delta y_1 \sim \exp\lambda_1 t, \quad \Delta x \sim \exp\lambda_1 t, \quad \Delta\mu \sim \exp\lambda_1 t \quad \text{or} \quad \exp\lambda_3 t \quad (c_2 = 0)$$

Then $\mu_1 \sim \exp 2\lambda_1 t$ or $\exp\,(\lambda_1 + \lambda_3)t$, hence $\mu_1 \to 0$ as $t \to -\infty$. Thus we see that, if the singularity of (1.1) is a node then $\mu_1 \to \infty$ on approaching the singularity, while in the case of a saddle point we have $\mu_1 \to 0$.

Let us now consider the term of (2.4) outside the integral sign, when $x = x^*$.

Since the singular point of (1.1) lies on a $(n-1)$-dimensional surface, δy_{j*} $(j = 2,\ldots,n)$ are arbitrary and consequently $\mu_j\,(x^*_-) = \mu_j\,(x^*_+)$ $(j = 2, \ldots, n)$.

We shall take a small area around the singularity of (1.1) and draw within it a straight line perpendicular to the x-axis and lying in the plane containing all integral curves of the system (1.1). Then, in the case of a node, this straight line will be intersected by a set of integral curves passing through the singularity and the quantity y_1 will vary within the limits $0 < |y_{1\pm}| \leqslant |\lambda_2|\,\varepsilon$, where ε is the radius of the area and λ_2 is given by (1.4) where the root should be taken with the minus sign. Consequently, if within the ε-neighborhood of the singularity $0 < |y_{1\pm}| < |\lambda_2|\,\varepsilon$, then $\delta y_{1\pm}{}^*$ is arbitrary and

$$\mu_1\,(x_-{}^*) = \mu_1\,(x_+{}^*) = 0 \tag{3.5}$$

i.e. the Lagrange multipliers should be continuous when passing through the singularity. Condition (3.5) supplements the conditions for integration of the system (2.5), when the Eqs. (1.1) have the node as a singularity.

If y_1 reaches its limit value within the ε-neighborhood, then the supplementary condition for integration of (2.5) will be

$$y_1 (x^* \pm \varepsilon) = \mp |\lambda_2| \varepsilon \tag{3.6}$$

In this case the quantities $\mu_1 (x_{\pm}{}^*)$ will be arbitrary. When (3.6) holds, it is advisable to take into consideration the flows with shock waves or with y_1 of constant sign, since an extremum may occur within this class of flows.

If the singularity defined by (1.2) is a saddle point, we have one integral curve with $y_1' (x^*) > 0$ and one with $y_1' (x^*) < 0$ passing through the singularity, and we then have $\delta y_1 \pm {}^* = 0$. Consequently the quantities $\mu_1 (x_-{}^*)$ and $\mu_1 (x_+{}^*)$ are arbitrary. We know however, from actual analysis, that $\mu_1 \to 0$ as $x \to x^*$ when the singularity is a saddle point. Thus the condition of continuity of the Lagrange multipliers is, in this case, satisfied automatically.

Let us now assume that the optimal controls suffer a discontinuity at the point at which y_1 vanishes. In this case the boundary conditions for integrating (2.5) can be chosen analogously to the case of the saddle point type singularity, since two integral curves of (1.1) pass through $x = x^*$ where $y_1 = 0$, and they have $y_1' \to \pm\infty$ as $y_1 \to 0$. From (2.5) it follows that, when $y_1 \to 0$ $(x \to x^*)$,

$$\mu_1 = c\ (x - x^*)^{1/2} \tag{3.7}$$

where c is an arbitrary constant. Expression (3.7) was obtained taking into account the fact, that

$$(y_1)^2 = 2f\,(x_{\pm}{}^*)\,(x - x^*)$$

as $x \to x^*$.

Thus $\mu_1 \to 0$ as $x \to x^*$, i.e. the Lagrange multipliers are in this case continuous.

4. In agreement with the choice of Lagrange multipliers, the first variation is

$$\delta I = \delta I = \int_a^b \left\{\left(\Phi_{v_i} - \frac{\mu_1 f_{v_i}}{y_1} - \mu_j g_{jv_i}\right)\delta v_i\right\} dx + \{\Phi (x_{d_-}) - \Phi (x_{d_+}) - \tag{4.1}$$

$$- \mu_s (x_d)\,[y_s' (x_{d_-}) - y_s' (x_{d_+})]\}\,\delta x_d \qquad (i = 1, \ldots, m;\ j = 2, \ldots, n;\ s = 1, \ldots, n)$$

Necessary conditions for an extremum can be obtained by the usual methods, with (2.2) taken into account.

If the controls are optimal, then any admissible variation will only lead to the decrease in the value of J, i.e. $\delta J \leq 0$. Assuming that the admissible variations δv_i can have any sign within the intervals $|v_i| < V_i$, negative on $v_i = + V_i$ and positive on $v_i = - V_i$, we obtain the necessary conditions for the maximum of J,

$$\begin{aligned} &F_i \equiv \Phi_{v_i} - \mu_1 f_{v_i} / y_1 - \mu_j g_{jv_i} = 0 \qquad (|v_i| < V_i) \\ &F_i \geqslant 0 \quad (v_i = V_i), \qquad F_i \leqslant 0 \quad (v_i = - V_i) \end{aligned} \qquad \begin{pmatrix} i = 1, \ldots, m \\ j = 2, \ldots, n \end{pmatrix} \tag{4.2}$$

and these conditions define the optimal controls.

Position of the point d at which the controls suffer a discontinuity, is given by the condition

$$\Phi (x_{d-}) - \Phi (x_{d+}) - \mu_s (x_d)[y'_s (x_{d-}) - y'_s (x_{d+})] = 0 \qquad (s = 1, \ldots, n)$$

Other terms outside the integral may appear in (4.1). They will be the outcome of the fact that some of the quantities y_{ka} and y_{kb} as well as the length of the interval of integration, may serve as controls. Optimal conditions for y_{ka} and y_{kb} will replace some of the conditions (1.6) as boundary conditions for (1.1), and the optimal conditions at a and b will define the position of the ends of the interval of integration.

5. As an example, we shall consider the problem of optimizing the flow in the channel of a magnetohydrodynamic electric power generator. The statement and the solution of this problem for subsonic and supersonic flows and the flows with shock waves, are given in [4 and 5]. Here we solve the problem under the assumptions of [4], but the flows considered will be transonic and passing through the velocity of sound at a singularity of the nodal type. Qualitative analysis [3] and numerical computation have shown, that such a flow is possible in a diverging channel when the interaction parameter $\Delta \geq 1.5$. Here $\Delta = \sigma_s{}^\circ B_m{}^\circ l^\circ / (\rho_s{}^\circ \sqrt{2h_s{}^\circ})$ where ρ_s^0 is the density, h_s^0 is the enthalpy, σ_s^0 is the conductivity of gas in the receiver while B_m^0 and l^0 represent the typical magnetic field and the typical length respectively.

One-dimensional steady flow of a nonviscous, non-heat-conducting perfect gas whose electrical conductivity is σ^0 in a plane channel of height $2y^0$ and in the presence of an external magnetic field B^0 (magnetic Reynolds numbers are assumed low), can be described by

$$\begin{aligned} &L_1 \equiv \rho u u' + p' + \Delta\sigma B\left(uB - \frac{\varphi}{y}\right) = 0 \\ &L_2 \equiv \left[uy\left(\frac{\varkappa}{\varkappa-1}p + \frac{\rho u^2}{2}\right)\right]' + \Delta\sigma\varphi\left(uB - \frac{\varphi}{y}\right) = 0 \qquad (5.1) \\ &L_3 \equiv (\rho u y)' = 0 \end{aligned}$$

Notation used in [4] is adopted here. Let us put $\sigma \equiv 1$. Then (5.1) can be reduced to the form

$$\begin{aligned} \frac{dM}{dx} &= \frac{M}{2}\,\frac{2+(\varkappa-1)M^2}{1-M^2}\times \qquad (5.2) \\ &\times\left[\frac{\varkappa M^2 y}{cu^2}\,\Delta\left(uB-\frac{\varphi}{y}\right)\left(uB - \frac{\varkappa-1}{\varkappa}\,\frac{1+\varkappa M^2}{2+(\varkappa-1)M^2}\,\frac{\varphi}{y}\right) - \frac{y'}{y}\right] \\ \frac{du}{dx} &= \frac{uM'}{M\,[1+{}^1/_2(\varkappa-1)M^2]} - \frac{(\varkappa-1)\,\Delta\varphi M^2\,(uB-\varphi/y)}{cu\,[2+(\varkappa-1)M^2]} \qquad (c=\rho u y) \end{aligned}$$

which is more suitable for obtaining a numerical solution of the problem and where M denotes the Mach number.

Singular points of (5.2) in the xuM-space lie on the plane $M = 1$ and on the line given by

$$\varkappa y\Delta\left(uB - \frac{\varphi}{y}\right)\left(uB - \frac{\varkappa-1}{\varkappa}\,\frac{\varphi}{y}\right) - \frac{cu^2 y'}{y} = 0, \qquad M = 1$$

When $\Delta \geq 1.5$, one of the branches of this line consists of the nodes with negative characteristic directions [3].

Assuming that the flow is supersonic at the inlet and that the inlet cross section y_a^0 of the channel is fixed, we can determine u_a and M_a at $x = 0$, by considering the flow for $x < 0$ and assuming it known. The quantity u_a (or M_a) can, in this case, be assigned arbitrary values within some limits imposed by the change of the form of the channel when $x < 0$. Under the open cycle working condition of the generator, the pressure p_∞ of the medium into which the fluid emerges, is assumed known at the outlet cross section. Thus we have, for (5.2),

$$u = u_a, \quad M = M_a \quad (x = 0), \qquad p = cu / \varkappa M^2 y = p_\infty \quad (x = x_b) \qquad (5.3)$$

The functions and parameters $y(x)$, $\varphi(x)$, $B(x)$, u_a, p_∞ and x_b the length of the channel act as controls.

The power available from the unit width of the generator

$$N = \int_0^{x_b} \Delta\varphi\left(uB - \frac{\varphi}{y}\right)dx$$

can be taken as the optimizing functional, and the optimal controls should satisfy the constraints formulated in [4] (see Formulas (1.3), (1.6) and (1.8)).

We shall write the auxilliary functional [4] in the form

$$I = \int_0^{x_b} \left[\Delta\varphi\left(uB - \frac{\varphi}{y}\right) + \mu_1 L_1 + \mu_2 L_2 + \mu_3 L_3\right] dx$$

Necessary conditions for the maximum of I obtained in [4] for $\sigma \equiv 1$, are valid for the present problem. The latter differs from the earlier problems only in the formulation of the boundary conditions for (5.2) and in the equations used to obtain the Lagrange multipliers.

Since the singular point of (5.2) is a node, then for the integration of the system

$$\frac{d\mu_2}{dx} = \frac{[\Delta B(\mu_1 B + \mu_2\varphi + \varphi) + \mu_1 cy'/y^2](\varkappa - 1)M^2}{(1 - M^2)\,cu}, \quad \frac{d\mu_1}{dx} = -\frac{\varkappa}{\varkappa - 1} uy \frac{d\mu_2}{dx} \tag{5.4}$$

one of the boundary condition is obtained from the requirement that the Lagrange multipliers are continuous at the singular point

$$\Delta B(\mu_1 B + \mu_2\varphi + \varphi) + \mu_1 cy'/y^2 = 0 \qquad (x = x^*) \tag{5.5}$$

while the second condition is obtained, as in [4], by considering the terms outside the integral sign for $x = x_b$, in the expression for δI

$$\mu_1 = -(\varkappa/\varkappa - 1)\,uy\mu_2 \qquad (x = x_b) \tag{5.6}$$

Factor μ_3 is not substantial in the present problem.

Optimal values of $B(x)$, $y(x)$, $\varphi =$ const, p_∞ and x_b were found by numerical methods for $\sigma \equiv 1$, $\varkappa = 5/3$, and $1.5 \leq \Delta \leq 2$. The largest admissible angle of inclination of the channel wall to the x-axis was chosen as 20°, the ratio $l^\circ/y_a^\circ = 10$ and the maximum height of the channel was $Y = 4.64$. No constraints were imposed on φ. Velocity of the gas at the inlet was assumed equal to the sound velocity.

Optimal value of $\varphi =$ const can be found from the formula (3.12) of [4], or from the equivalent differential equation

$$d\chi/dx = \Delta\,[(1 + \mu_2)(uB - \varphi/y) - (\mu_1 B + \mu_2\varphi + \varphi)\,y^{-1}] \tag{5.7}$$

with the boundary conditions

$$\chi(0) = \chi(x_b) = 0 \tag{5.8}$$

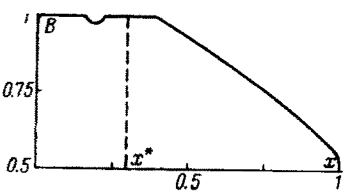

Fig. 1

Choice of φ leads to fulfilment of one of these conditions. Thus we must solve the boundary value problem for five differential Eqs. (5.2), (5.4) and (5.7) with the following boundary conditions: (5.3), (5.5), (5.6) and (5.8).

The equations were integrated by the Runge – Kutta method from $x = 0$ to $x = x^*$ and from $x = x_b$ to $x = x^*$, and x was taken as the independent variable only when $|M'| < 1$. When $|M'| \geq 1$, then M was used as the independent variable.

The lacting boundary conditions $\mu_1(0)$ and $\mu_2(0)$ required when integrating from 0 to x^* and $\mu_1(x_b)$ with $\mu_2(x_b)$ when integrating from x_b to x^* as well as φ, were selected by means of approximations with respect to five parameters and using the Newton's method.

Equations (5.4) were linearized in the small neighbourhood of the singular point, and the choice of this particular method of integration was dictated by the necessity of obtaining a stable solution near the node.

Computations have shown that $y(x) = kx + 1$, $(k = \text{tg}\,20° l^\circ / y_a^\circ)$, and $x_b = 1$ are optimal, while optimal $B(x)$ consists of the segments of the boundary extremum $B = 1$

together with the segments of the two-sided extremum. Fig. 1 shows the optimal $B(x)$ for $\Delta = 2$.

We have also computed, for comparison, the power of the generator at $B = 1$ with the remaining optimal parameters. We found, that the power of the generator at the optimal value of $B(x)$ exceeds that at $B = 1$ ($\Delta = 2$) by 8.3 %.

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